Advanced Topics in Probability - Lecture 12

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Continuing Long-Range Order for Spin $O(n)$ models in $d \geq 3$ at low temperatures

Let L be an even number. Denote $\Lambda := \Lambda_L = \left\{-\frac{L}{2}+1, \dots \frac{L}{2}\right\}^d$ the d-dimensional discrete torus of side length L, and define the Shifted Partition Function as:

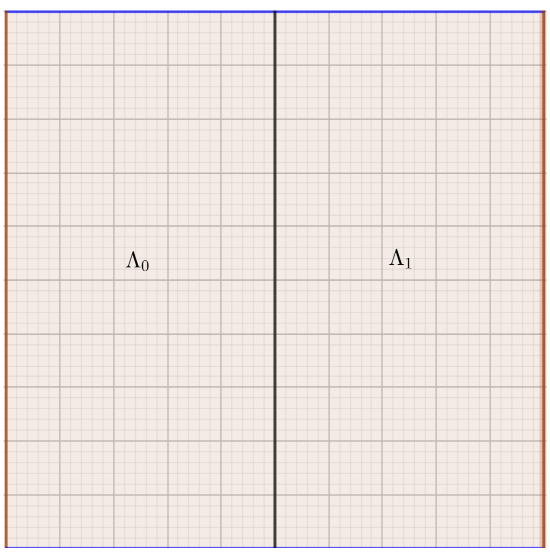
$$Z(f) = \int_{\Omega_\Lambda} \exp\left(-\beta \sum_{\substack{u \sim v \\ u,v \in V}} \|\sigma_u - \sigma_v + f_u e_1 - f_v e_1\|_2^2\right) \mathrm{d}\,\sigma$$

where $\Omega_\Lambda = \left\{\sigma : \Lambda \to S^{n-1}\right\}$, $f : \Lambda \to \mathbb{R}$, and where $\mathrm{d}\,\sigma = \prod_{v \in \Lambda} \mathrm{d}\,m(\sigma_v)$, and m is the Lebesgue measure on S^{n-1}.

Definition. Gaussian Domination (GD) is said to occur if and only if for all $f : \Lambda \to \mathbb{R}$, $Z(0) \geq Z(f)$.

We will prove GD for Ω_Λ using reflection positivity. **Open problem:** find robust proofs for GD that will work in other domains.

Reflection Positivity



Let Λ_0, Λ_1 be the two halves of Λ, split at the first coordinate. Let $R : \Lambda \to \Lambda$ be the reflection mapping Λ_0 to Λ_1 and vice versa, i.e. $R(x_1, x_2, \dots, x_d) = (1 - x_1, x_2, \dots, x_d)$, and for functions $f : \Lambda_0 \to \mathbb{R}$ define $(Rf)(x) = f(Rx)$, and similarly for $f : \Lambda_1 \to \mathbb{R}$. For $f : \Lambda \to \mathbb{R}$, define $f_0 = f\restriction_{\Lambda_0}$ and $f_1 = f\restriction_{\Lambda_1}$, and write $Z(f) = Z(f_0, f_1)$.

Definition. Z is said to be reflection positive if for all f_0, f_1 as above:

$$Z(f_0, f_1) \leq \sqrt{Z(f_0, Rf_0)\, Z(Rf_1, f_1)}$$

Proposition. *[Reflection positivity] Z is reflection positive.*

Remark. Reflection positivity is a more general technique, sometimes used with cuts going through vertices instead of edges, whose main consequences are:

- The infra-red bound
- The chessboard estimate

(See lecture notes of Peled-Spinka/Biskup).

Proof of Gaussian Domination from Proposition

Proof. First, note that if the difference in f along an edge is large, then $Z(f)$ will be small. Thus, maximizers of $Z(\cdot)$ exist, since one can look for them in a compact set (noting that $Z(f+c)=Z(f)$ for constant c). Let $\overline{f}$ be a maximizer of Z, which also minimizes $k(f):=\#\{\{u,v\}\in E \mid f_u\neq f_v\}$. We wish to show that $k(\overline{f})=0$. Indeed, suppose $k\geq 1$, and let $e=\{u,v\}$ be an edge such that $\overline{f}_u\neq\overline{f}_v$. By rotating and translating, one may assume that e connects Λ_0 and Λ_1. Now by the proposition:

$$Z\left(\overline{f}_0,\overline{f}_1\right)\leq\sqrt{Z\left(\overline{f}_0,R\overline{f}_0\right)Z\left(R\overline{f}_1,\overline{f}_1\right)}.$$

Thus, since $\overline{f}$ is a maximizer of Z, so are $\left(\overline{f}_0,R\overline{f}_0\right)$, $\left(R\overline{f}_1,\overline{f}_1\right)$. Now, note that $\frac{1}{2}\left(k\left(\overline{f}_0,R\overline{f}_0\right)+k\left(R\overline{f}_1,\overline{f}_1\right)\right)<k\left(\overline{f}_0,\overline{f}_1\right)$, since on the boundaries between Λ_0,Λ_1, both $\left(\overline{f}_0,R\overline{f}_0\right)$ and $\left(R\overline{f}_1,\overline{f}_1\right)$ agree. So one of $k\left(\overline{f}_0,R\overline{f}_0\right),k\left(R\overline{f}_1,\overline{f}_1\right)$ is smaller than $k(f)$, and thus $\overline{f}$ is not the minimal maximizer. □

Proof of Reflection Positivity

Proof. Two tricks will be used here:

1. The first trick has several names:
   - Fourier transform of the Gaussian distribution
   - Habbard-Stratonovich transformation
   - Introduce a complex field to decouple the interaction

$$\forall a\in\mathbb{R}.\ \exp\left(-\tfrac{1}{2}a^2\right)=\int_{-\infty}^{\infty}\overbrace{\frac{\mathrm{d}\xi}{\sqrt{2\pi}}e^{-\frac{1}{2}\xi^2}}^{\text{non-negative measure}}\overbrace{e^{i\xi a}}^{\text{Linear in }a}.$$

2. Cauchy-Schwarz inequality.

Then, letting $f=(f_0,f_1)$,

$$Z(f_0,f_1)=\int_{\Omega_\Lambda}\overbrace{\prod_{\substack{u\sim v\\u,v\in\Lambda_0}}e^{-\beta\|\sigma_u-\sigma_v+f_ue_1-f_ve_1\|_2^2}}^{h_0}\overbrace{\prod_{\substack{u\sim v\\u,v\in\Lambda_1}}e^{-\beta\|\sigma_u-\sigma_v+f_ue_1-f_ve_1\|_2^2}}^{h_1}\overbrace{\prod_{\substack{u\sim v\\u\in\Lambda_0,v\in\Lambda_1}}e^{-\beta\|\sigma_u-\sigma_v+f_ue_1-f_ve_1\|_2^2}}^{\text{cut edges}}\mathrm{d}\,\sigma.$$

Using trick #1 in the cut edges:

$$=\int_{\Omega_\Lambda}\mathrm{d}\,\sigma h_0h_1\prod_{\substack{u\sim v\\u\in\Lambda_0,v\in\Lambda_1}}\prod_{j=1}^{n}\int_{-\infty}^{\infty}\frac{\mathrm{d}\,\xi_{u,v}^j}{\sqrt{2\pi}}e^{-\frac{1}{2}\left(\xi_{u,v}^j\right)^2}e^{i\xi_{u,v}^j\sqrt{2\beta}(\sigma_{u,j}-\sigma_{v,j}+f_ue_{1,j}-f_ve_{1,j})}$$

$$=\int d\mu(\xi)\int_{\Omega_{\Lambda_0}}h_0\prod_{u\text{ on }\Lambda_0\text{'s boundary}}e^{i\xi_{u,v}^j\sqrt{2\beta}(\sigma_{u,j}+f_ue_{1,j})}\mathrm{d}\,\sigma_0\int_{\Omega_{\Lambda_1}}h_1\prod_{v\text{ on }\Lambda_1\text{'s boundary}}e^{-i\xi_{u,v}^j\sqrt{2\beta}(\sigma_{v,j}+f_ve_{1,j})}\mathrm{d}\,\sigma_1$$

where

$$\mathrm{d}\,\mu(\xi)=\prod_{\substack{u\sim v\\u\in\Lambda_0,v\in\Lambda_1}}\prod_{j=1}^{n}\frac{\mathrm{d}\,\xi_{u,v}^j}{\sqrt{2\pi}}e^{-\frac{1}{2}\left(\xi_{u,v}^j\right)^2}$$

is a non-negative measure. Using Cauchy-Schwarz:

$$Z(f_0,f_1)\leq$$

$$\sqrt{\int d\mu(\xi)\left|\int_{\Omega_{\Lambda_0}}h_0\prod_{u\text{ on}\Lambda_0\text{'s boundary}}e^{i\xi_{u,v}^j\sqrt{2\beta}(\sigma_{u,j}+f_ue_{1,j})}\mathrm{d}\,\sigma_0\right|^2}\sqrt{\int d\mu(\xi)\int_{\Omega_{\Lambda_1}}\left|h_1\prod_{v\text{ on}\Lambda_1\text{'s boundary}}e^{-i\xi_{u,v}^j\sqrt{2\beta}(\sigma_{v,j}+f_ve_{1,j})}\mathrm{d}\,\sigma_1\right|^2}$$

And now, remember that $|z|^2=z\overline{z}$, and

$$\overline{\int_{\Omega_{\Lambda_0}}h_0\prod_{u\text{ on }\Lambda_0\text{'s boundary}}e^{i\xi_{u,v}^j\sqrt{2\beta}(\sigma_{u,j}+f_ue_{1,j})}\mathrm{d}\,\sigma_0}=\int_{\Omega_{\Lambda_1}}h_0(Rf_0)\prod_{v\text{ on }\Lambda_1\text{'s boundary}}e^{-i\xi_{u,v}^j\sqrt{2\beta}\left(\sigma_{v,j}+(Rf)_ve_{1,j}\right)}\mathrm{d}\,\sigma_1$$

as a reflection, and similarly for f_1, and so $Z(f_0,f_1)\leq\sqrt{Z(f_0,Rf_0)Z(Rf_1,f_1)}$. □

Disordered Spin Systems

Lattice spin systems in a random environment.

Examples

1. Random-Field Ising Model (RFIM): $\sigma : \Lambda \to \{-1, 1\}$,

$$H(\sigma) = H^{\eta}(\sigma) := -\sum_{u\sim v} \sigma_u \sigma_v - \lambda \sum_v \eta_v \sigma_v$$

   where $\lambda > 0$ is a parameter governing the strength of the disorder, $(\eta_v)_{v\in\mathbb{Z}^d}$ IID, $\mathbb{E}\eta_0 = 0$, $\text{Var}(\eta_0) = 1$. E.g., (η_v) are IID $\mathcal{N}(0,1)$ or IID $\frac{\delta_1+\delta_{-1}}{2}$. η is the environment. For each fixed value of η, we have an Ising model, with "apriori tendencies" of the spins to follow the signs of η and with λ controlling the relative strength of the neighbours' effect vs. the apriori tendency. λ is called the "disorder strength".

2. Random-Field Potts Model (RFPM) with q states: $\sigma : \Lambda \to \{1, \ldots, q\}$,

$$H(\sigma) = -\sum_{u\sim v} \mathbb{1}_{\sigma_u=\sigma_v} - \lambda \sum_v \sum_{j=1}^{q} \eta_{v,j} \mathbb{1}_{\sigma_v=j}$$

   where $\eta : \mathbb{Z}^d \times \{1, \ldots, q\} \to \mathbb{R}$ IID as before.

3. Random-Field Spin $O(n)$ model, $n \geq 2$: $\sigma : \Lambda \to S^{n-1}$

$$H(\sigma) = -\sum_{u\sim v} \sigma_u \cdot \sigma_v - \lambda \sum_v \eta_v \cdot \sigma_v$$

   η IID taking values in $\mathbb{R}^n$, e.g. $\mathcal{N}(0, I_n)$. We write $\cdot$ for standard inner product in $\mathbb{R}^n$.

4. Disordered Ferromagnet and Edwards-Anderson Spin Glasses: $\sigma : \mathbb{Z}^d \to \{-1, 1\}$

$$H(\sigma) = -\sum_{u\sim v} \eta_{u,v} \sigma_u \sigma_v$$

   $\eta_{u,v}$ IID.

   - Disordered Ferromagnet: $\eta \geq 0$.
   - Spin Glasses: η is both positive and negative.

In all the examples above, form a probability measure in a finite volume Λ with boundary conditions τ, by fixing $\sigma\restriction_{\Lambda^c} = \tau\restriction_{\Lambda^c}$ and setting the density proportional to $\exp\left(-\beta H_{\Lambda}^{\tau,\eta}(\sigma)\right)$ with β= inverse temperature.

Quenched: Write $\langle\cdot\rangle_{\Lambda}^{\tau,\eta} = \langle\cdot\rangle^{\tau}$ for expectation in the above measure.

Averaged: We use $\mathbb{P}$ and $\mathbb{E}$ for averages over η.

Ground State: The case of zero temp. $(\beta = \infty)$ corresponds to a uniform distribution over energy minimizing configurations. We will talk of cases where there is a unique such configuration (in finite volume) and denote this configuration by $\sigma^{\Lambda,\eta,\tau} = \sigma^{\tau}$.

Understanding the ground state is usually the main challenge in understanding the low-temperature behaviour.

Random-Field models were first analyzed by Imry-Ma (1975):

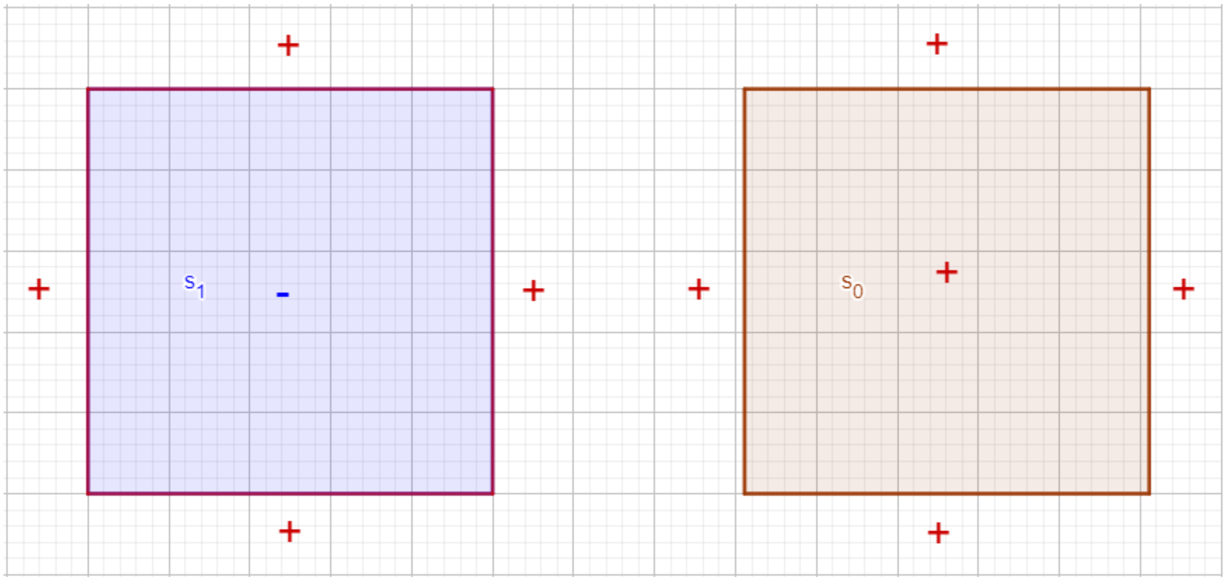
Imry-Ma Phenomenon

Random-Field Spin Models do not have an ordered phase in low dimensions.

- d=2: All such models are not ordered!
- $2 \leq d \leq 4$: Random-Field Spin $O(n)$ models with $n \geq 2$ with $O(n)$-invariant η are disordered.
- $d \geq 3$: RFIM, RFPM have low temp. and small λ ordered phase.

The last claim (regarding $d = 3$) was challenged by other physicists, but eventually proved true by mathematicians Imbrde (1985) and Bricmont-Kupiainen (1988).

Heuristic: RFIM with $(+)$ boundary: Is the configuration of all $(+)$ more likely than all $(-)$?

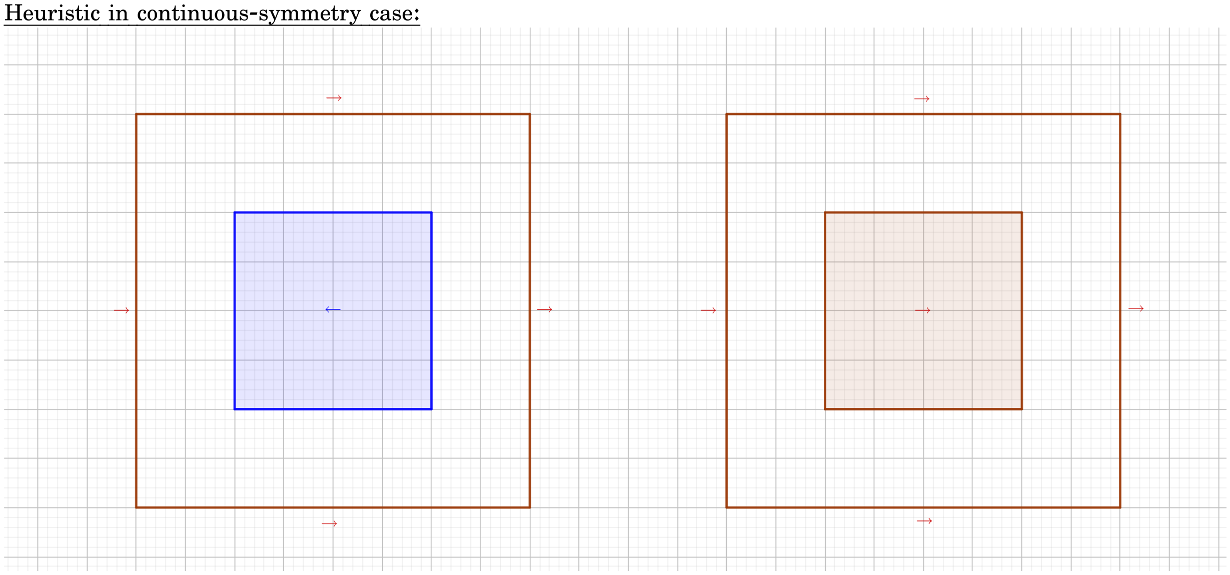


$$\Delta=H^{\eta}\left(s_{0}\right)-H^{\eta}\left(s_{1}\right)\approx\overbrace{-L^{d-1}}^{\text{bdry. conflict}}+\overbrace{\lambda\mathcal{N}\left(0,L^{d}\right)}^{\text{field}}$$

$\Delta<0$ means that the boundary wins. Is $L^{d-1}>\lambda\mathcal{N}\left(0,L^{d}\right)$?

Yes, when $d\geq3$; No, when $d=1$. In $d=2$ we have a constant $\left(\approx e^{-\frac{c}{\lambda^{2}}}\right)$ probability that the field wins, whence the field wins in some random sufficiently large box.

Heuristic in continuous-symmetry case:



the energetic cost $\Delta\approx\overbrace{-L^{d-2}}^{\text{bdry. conflict}}+\overbrace{\lambda\mathcal{N}\left(0,L^{d}\right)}^{\text{field}}$ is balanced in $d=4$.

Theorem. *[Aizenman-Wehr 1989, version here from Dario-Harel-Peled 2021]*

- *RFIM, RFPM in $d=2$:*

$$\forall0<\beta\leq\infty.\ \mathbb{E}\left[\sup_{\tau_{1},\tau_{2}}\left|\frac{1}{L^{2}}\sum_{v\in\Lambda_{L}}\left(\left\langle \mathbb{1}_{\sigma_{v}=j}\right\rangle _{\Lambda_{L}}^{\tau_{1}}-\left\langle \mathbb{1}_{\sigma_{v}=j}\right\rangle _{\Lambda_{L}}^{\tau_{2}}\right)\right|\right]\xrightarrow{L\to\infty}0$$

- *RF Spin $O\left(n\right),n\geq2,\ 2\leq d\leq4$, η rotationaly invariant:*

$$\mathbb{E}\left[\sup_{\tau_{1}}\left|\frac{1}{L^{2}}\sum_{v\in\Lambda_{L}}\left\langle \sigma_{v}\right\rangle _{\Lambda_{L}}^{\tau}\right|\right]\xrightarrow{L\to\infty}0$$

DHL showed that:

$$\mathbb{E}\left[\sup_{\tau_{1}}\left|\frac{1}{L^{2}}\sum_{v\in\Lambda_{L}}\left\langle \sigma_{v}\right\rangle _{\Lambda_{L}}^{\tau}\right|\right]\leq\begin{cases}c/L^{\frac{1}{3}} & d=2\\ c/L^{\frac{1}{5}} & d=3\\ \frac{c}{\sqrt{\log\log L}} & d=4\end{cases}$$

More is known for RFIM in $d = 2$ Aizenman-Wehr: By monotonicity ((+) boundary conditions implies more (+)s), there is no need to average over Λ_L:

$$m_L := \mathbb{E}\left[\langle\sigma_0\rangle^+_{\Lambda_L}\right] \xrightarrow{L\to\infty} 0$$

The rate of decay was refined until recently it was shown by Ding-Xia **2019** ($T = 0$ and then $T > 0$) and Aizenman-Harel-Peled, $m_L \le C_\lambda e^{-c_\lambda L}$.

Ding-Wirth (2020): For $T > 0$ and low temp., $d = 2$, boundary conditions lose their effect at $L \approx e^{\lambda^{-\frac{4}{3}+o(1)}}$ as $\lambda \downarrow 0$. Conjecturally, similar behaviour holds for other models, e.g. RFPM:

Conjecture. $\forall 0 < \beta \le \infty, \lambda > 0,\ d = 2 : \forall 1 \le j \le q$ *in RFPM:*

$$\mathbb{E}\left[\sup_{\tau_1,\tau_2}\left|\langle\sigma_v\rangle^{\tau_1}_{\Lambda_L} - \langle\sigma_v\rangle^{\tau_2}_{\Lambda_L}\right|\right] \xrightarrow{L\to\infty} 0 \tag{1}$$

(1) *is not known, even at* $\beta = \infty$*.*

Conjecture. *[Unique infinite volume ground state pair in $d = 2$ spin glass] In $d = 2$, there is a unique ground state pair in $\mathbb{Z}^2$. A finite-volume manifestation:* $\forall 0 < \beta \le \infty$*,*

$$\mathbb{E}\left[\sup_{\tau_1,\tau_2}\left|\left\langle\mathbb{1}_{\sigma_0\sigma_{e_1}=+1}\right\rangle^{\tau_1}_{\Lambda_L} - \left\langle\mathbb{1}_{\sigma_0\sigma_{e_1}=+1}\right\rangle^{\tau_1}_{\Lambda_L}\right|\right] \xrightarrow{L\to\infty} 0$$

Only a spatially-averaged version is known (Aizenman-Wehr, Dario-Harel-Peled).